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On the Local Stability of a Certain Class of Polynomial Differential System

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Abstract

We state and prove a condition for the local stability of a certain class of two dimensional system of polynomial differential equations. We give some examples of polynomial differential systems of equations to demonstrate that this local stability condition established for the trivial equilibrium point $(0, 0)$ is quite sharp and compare our result with the well known Lyapunov local stability criterion (Lyapunov's second method).

Keywords: equilibrium; stability; asymptotic stability; differential system.

1 Introduction

The study of systems of polynomial differential equations is directly related to the second part of the 16th Hilbert problem that was posed by David Hilbert at the international congress of mathematicians in 1900. In this study we consider a polynomial differential system given by

$$\dot{x} = y + A(x, y), \qquad \dot{y} = -x + B(x, y), \tag{1}$$

where $A(x, y)$ and $B(x, y)$ are polynomials that both vanish at the trivial equilibrium point $(0, 0)$ of degree at least two. Such classes of systems of polynomial differential equations are important in explaining/modelling several different phenomena in science for example, in Ecology, most models to explain community interaction of N species whose biomass is denoted with x_i take the general form $\dot{x}_i = x_i F_i(x)$ for all $i = 1, 2, 3, \ldots, N$. An example of such species interaction models is the prey-predator system with competition and/or cooperation that has been modelled and studied using two dimensional system of polynomial differential equations [1, 7], in electrical engineering, an electrical circuit consists of elements each of which has current, $I(t)$ flowing through and the voltage difference, $V(t)$. The relationship between $I(t)$ and $V(t)$ is modelled by equations resulting from Kirchoff's law that can be reduced to the form of Liénard's equations that take the form of (1), [15, 9]. One can also reduce certain partial differential equations that describe control problems to a system of ordinary differential equations and study the resulting system for example in [8] and [13].

The most common problem/question is to discuss the qualitative properties of the systems' equilibria, solutions and periodic orbits in their neighbourhood(s). This qualitative theory and application of the system of differential equations can be done by investigating the properties of solutions for the system of non-linear differential equations via for example establishing their stability and boundedness. In [17, 16], the authors use Lyapunov second method and other methods to investigate different non-linear differential systems for boundedness and asymptotic stability of solutions and asymptotic stability, integrability and boundedness. In particular, in [17] the authors consider the $X'' + a(t)X' + b(t)X = 0$ where $X \in \mathbb{R}^n, t \in \mathbb{R}^+, a(\cdot), b(\cdot) : \mathbb{R}^+ \to (0, \infty)$ are continuous functions both having lower and upper bounds and uses integral inequalities to prove some sufficient conditions for the existence of bounded and integrable solutions to the equation after converting it into a differential system.

The author in [16] considers the non-linear second order differential equation $\ddot{X} + F(t, X, \dot{X})\dot{X} + b(t)H(X) = 0$ where all the functions involved are chosen so as the differential equation has a solution and uses the Lyapunov second method to investigate the asymptotic stability, integrability and boundedness of its solutions. The interest in some other cases is in for instance existence of first integrals e.g [2, 5], others in the existence of limit cycles surrounding an equilibrium point e.g [6] provides necessary and sufficient conditions for the existence of a Hopf bifurcation for a certain class of a 2- dimensional polynomial differential system whose linear part has eigenvalues $\pm\omega i$ with $\omega \neq 0$ where $i = \sqrt{-1}$. [15] gives a necessary condition for local asymptotic stability of non-linear systems of the form (1) with exogenous disturbance. The asymptotic stability of Kolmogorov type systems have been studied by many researchers e.g. [14] uses Lyapunov functions and the Razumikhin technique to obtain a criteria for uniform global asymptotic stability of sets for delayed Kolmogorov type systems. In [19], the authors introduce some stronger forms of sensitivities in the dynamical systems of semigroup actions and obtain sufficient conditions for a certain dynamical systems to have such sensitivities and other studies relating to the Kolmogorov type system are contained in [4, 5]. In this paper, we use the idea of polar coordinates used in [1] to prove necessity of some results in the characterisation of center conditions for a cubic Kolmogorov differential system to prove in particular a local stability result stated as follows,

Theorem 1.1. *Suppose that the definite integral*

$$\int_0^{2\pi} \alpha_1(\theta)d\theta = -a,$$

for some $a > 0$. Then there exists a small $\epsilon_\star$ such that if we take the solution to the ordinary differential system (1) with initial conditions $\theta(0) = 0$ and $r(0) = \epsilon_\star$ then $\lim_{t\to\infty} r(t) = 0$.

The local stability result (Theorem 1.1) is a special case of Theorem 2 in section 3.4 of [11] however, the idea of our proof is new and involves checking what happens to $r(t)$ as $t \to \infty$ if one chooses an initial point $r(0)$ closer to the equilibrium point $(0, 0)$ which for our system is a center and rotates around it as $t \to \infty$.

2 Preliminaries

The following basic definitions and results can be found in basic texts of dynamical systems such as [10] and [11].

Let $B = \{\underline{x}(t) \in \mathbb{R}^n : \|\underline{x}(t)\| \le r\}$ for $r < \infty$, be an open ball in $\mathbb{R}^n$ and $\underline{x} = (x_1(t), x_2(t), \dots, x_n(t))$ a position vector at a time t in $\mathbb{R}^n$.

Definition 2.1. *The vector $\underline{x}$ is said to be an equilibrium point of the differential polynomial system $\dot{\underline{x}} = f(\underline{x})$ if $\dot{\underline{x}} = 0$.*

Theorem 2.1. (Theorem 4.1 in [10]) *Let $x^\star = 0$ be an equilibrium point of the ODE-system of the form (1) and if there exists a Lyapunov function $V(x) > 0$ for all $x(t) \in B$ such that $\dot{V}(x) < 0$ for all $x(t) \in B$, then the equilibrium is locally asymptotically stable.*

We note that Theorem 2.1 re-stated here is the Lyapunov second Theorem for local stability of equilibrium points however, there are other forms of the Lyapunov stability theorems for example the Lyapunov Theorem for exponential stability and that for uniform stability [3].

For this paper, we are interested in this version (Theorem 2.1) because we would like to compare our result that gives a condition on a certain integral to be satisfied for a polynomial differential system of the form (1) to have local asymptotic stability only in the neighbourhood of the origin as an equilibrium point of a differential polynomial system whose linearised form gives the eigenvalues $\pm\omega i$ with $\omega \neq 0$ where $i = \sqrt{-1}$. Our main idea is if we start at point on a path $r(t)$ very close to the trivial equilibrium point $(0, 0)$ what happens to the radius $r(t)$ as one moves/cycles around the equilibrium point as $t \to \infty$. We shall use the big O notation in our proof where big O as used will have the same meaning as that in Definition 2.2 below.

Definition 2.2. *For a given real variable x and a positive function $f(x)$, the $O(f(x))$ denotes the existence of some $g(x)$ such that there exists $K > 0$ and a neighbourhood of x, B_x such that $f(x) \le K|g(x)|\ \forall\, x \in B_x$.*

3 Local Stability of a Subclass of Polynomial Differential System of the form (1).

We state and prove below a basic result upon which the proof of our main result on local stability of the trivial equilibrium point $(0, 0)$ of the polynomial differential system (1) will rely,

Lemma 3.1. *Let $\alpha(\theta)$ be a general homogeneous cubic polynomial in both sine and cosine functions then its integral over the interval $[0, 2\pi]$ vanishes i.e. $\int_0^{2\pi} \alpha(\theta) d\theta = 0$.*

Proof. Let the general homogeneous cubic polynomial in both sine and cosine functions be given by $\alpha(\theta) = a \sin^3 \theta + b \cos^3 \theta$ where a, b are constants, then

$$\int_0^{2\pi} \alpha(\theta) d\theta = \int_0^{2\pi} (a \sin^3 \theta + b \cos^3 \theta) d\theta = 0.$$

□

Now consider a differential polynomial system of the form defined in (1) reproduced here as

$$\dot{x} = y + A(x, y), \qquad \dot{y} = -x + B(x, y), \tag{2}$$

where $A(x, y)$ and $B(x, y)$ are polynomials which both vanish at the origin of degree at least two and that $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$ with t is an independent variable. In polar coordinates (r, θ), with $x = r \cdot \cos\theta$ and $y = r \cdot \sin\theta$ the Kolmogorov-type differential polynomial system (2) becomes

$$\dot{r} = \cos\theta \cdot A(r \cdot \cos\theta, r \cdot \sin\theta) + \sin\theta \cdot B(r \cos\theta, r \cdot \sin\theta), \tag{3}$$

$$\dot{\theta} = 1 + \frac{\cos\theta \cdot B(r \cdot \cos\theta, r \cdot \sin\theta) - \sin\theta \cdot A(r \cos\theta, r \cdot \sin\theta)}{r}. \tag{4}$$

Since $A(x, y)$ and $B(x, y)$ are homogeneous polynomials, the right hand side of (3) has an expansion in terms of θ and r of the form,

$$r^2 \cdot \alpha_0(\theta) + r^3 \cdot \alpha_1(\theta) + \cdots.$$

We further note that each coefficient function of $r, \alpha_k(\theta)$ and their corresponding powers is a homogeneous polynomial in the sine and cosine functions of degree $k + 3$ for every $k \geq 0$. Similarly the right hand side of (4) has an expansion in terms of θ and r of the form,

$$1 + r \cdot \beta_0(\theta) + r^2 \cdot \beta_1(\theta) + \cdots.$$

From where, $\beta_k(\theta)$ is a homogeneous polynomial in the sine and cosine functions of degree $k + 3$ for every $k \geq 0$. With these, we state our main result in this section as,

Theorem 3.1. *Suppose that the integral*

$$\int_0^{2\pi} \alpha_1(\theta) d\theta = -a,$$

for some $a > 0$. Then there exists a small $\epsilon_\star$ such that if we take the solution to the Kolmogorov-type ODE - system (2) with the initial conditions $\theta(0) = 0$ and $r(0) = \epsilon_\star$ then $\lim_{t\to\infty} r(t) = 0$.

Proof. Considering the expansions of the right hand sides of (3) and (4) and that $A(x, y)$ and $B(x, y)$ are polynomial functions then, the function $r(\theta)$ satisfies the first order differential equation,

$$\frac{dr}{d\theta} = r^2 \cdot \frac{\alpha_0(\theta) + r \cdot \alpha_1(\theta) + \cdots}{1 + r \cdot \beta_0(\theta) + r^2 \cdot \beta_1(\theta) + \cdots}. \tag{5}$$

Starting with a small $\epsilon_\star > 0$ and the initial condition $r(0) = \epsilon_\star$, we get a unique solution $r(\theta)$ to this ODE (5). The given local asymptotic stability condition is deduced from the fact that,

$$\int_0^{2\pi} \left(\frac{dr}{d\theta}\right) < 0. \tag{6}$$

To show (6) we write $r(\theta) = \epsilon_\star + \rho(\theta)$ where $\rho(0) = 0$ and since we consider a rotation through $360°$ around and closer to the origin then it is enough to pursue the solution when $0 \le \theta \le 2\pi$. From (5) the derivative of $\rho(\theta)$,

$$\frac{d\rho}{d\theta} = O(\epsilon_\star^2), \tag{7}$$

where the last term in (7) refers to the big-O. In particular there exists a constant C which is independent of $\epsilon_\star$ as soon as it is small enough, such that the ρ-function (which depends on the initial choice of $\epsilon_\star$) satisfies the estimate

$$|\rho(\theta)| \le C\epsilon_\star^2 : \ 0 \le \theta \le 2\pi. \tag{8}$$

Next, $r^2(\theta) = \epsilon_\star^2 + 2\epsilon_\star\rho(\theta) + \rho^2(\theta)$ and that $\frac{dr}{d\theta} = \frac{d\rho}{d\theta}$ then (5) entails that

$$\frac{d\rho}{d\theta} = \epsilon_\star^2 \cdot a_0(\theta) + O(\epsilon_\star^3), \tag{9}$$

where $a_0(\theta)$ is a cubic polynomial in sine and cosine. Its integral over a small cycle around the origin that is $0 \le \theta \le 2\pi$ vanishes by Lemma 3.1 and therefore

$$\rho(2\pi) = O(\epsilon_\star^3). \tag{10}$$

Now, since $\beta_0(\theta)$ is also a cubic polynomial in sine and cosine then its integral over the interval $[0, 2\pi]$ vanishes by Lemma 3.1. It can be deduced from (7), (8), (9) and $r^2(\theta) = \epsilon_\star^2 + 2\epsilon_\star\rho(\theta) + \rho^2(\theta)$ that,

$$\int_0^{2\pi} \frac{dr}{d\theta} d\theta = \epsilon_\star^3 \int_0^{2\pi} \alpha_1(\theta) d\theta + 2\epsilon_\star \int_0^{2\pi} a_0(\theta) d\theta + O(\epsilon_\star^4). \tag{11}$$

From (9) and (10)

$$2\epsilon_\star \int_0^{2\pi} a_0(\theta) d\theta = \frac{2}{\epsilon_\star} \int_0^{2\pi} \rho'(\theta)\rho(\theta) d\theta + O(\epsilon_\star^4) = O(\epsilon_\star^5). \tag{12}$$

From (11) we conclude that if $\epsilon_\star$ is sufficiently small and we choose a constant $a > 0$ then

$$r(2\pi) \le r(0) - \epsilon_\star^3 \cdot a + O(\epsilon_\star^4). \tag{13}$$

The estimate (13) completes the proof of Theorem 3.1. □

4 Examples Demonstrating and Comparing Our Local Stability Result (Theorem 3.1) and Lyapunov's Second Method for Local Stability (Theorem 2.1)

Example 4.1. *Consider the system*

$$\dot{x} = y - x^3, \qquad \dot{y} = -x - y^3. \tag{14}$$

The ODE system (14) *is in the form* (1) *with homogeneous polynomials that vanish at the origin of degree* 3 *given by* $A(x, y) = -x^3$ *and* $B(x, y) = -y^3$. *The linearised system at the equilibrium point* $(0, 0)$ *has the eigenvalues* $\lambda = \pm i$. *We describe local stability of the trivial equilibrium point* $(0, 0)$ *using the two results mentioned in the section title of Section* 4. *The phase portrait for the system* (14) *is as shown in Figure* 1 *below;*

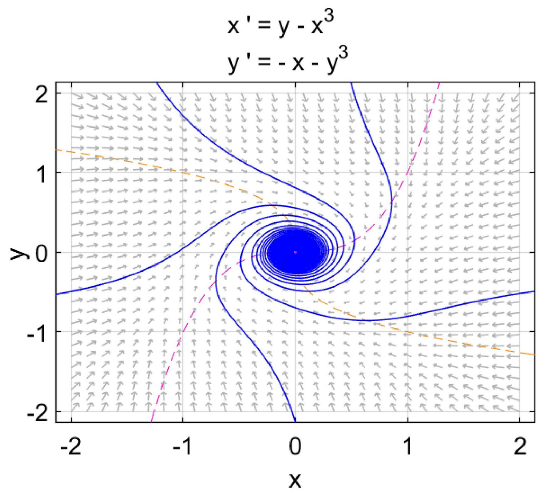

Figure 1: Phase portrait for system (14).

(*i*) *Lyapunov's second Method* (*Theorem* 2.1)*:*
Consider the Lyapunov function $V(x, y) = x^2 + y^2$ *where* $(x, y) \in \Re^2$.
Now $\frac{d}{dt}(V) = \partial_x(V(x, y)) \cdot \dot{x} + \partial_y(V(x, y)) \cdot \dot{y} = -2(x^4 + y^4) < 0$ *for all* $(x, y) \neq (0, 0)$. *From Theorem* 2.1 *and the derivative of* $V(x, y)$ *with respect to* t *being negative, it implies that the value of* V *decreases along the trajectories near the origin as shown in Figure* 1 *thus the equilibrium point* $(0, 0)$ *is locally asymptotically stable.*

(*ii*) *Our method* (*Theorem* 3.1)*:*
Let $x = r \cdot \cos\theta$ *and* $y = r \cdot \sin\theta$. *Here after expansions* $\alpha_1(\theta) = -(\cos^4\theta + \sin^4\theta)$ *and* $\int_0^{2\pi} \alpha_1(\theta) d\theta = -\frac{3}{2}\pi < 0$, *thus the equilibrium point* $(0, 0)$ *is locally asymptotically stable since it satisfies the local stability condition of Theorem* 3.1 *with a non-negative* $a = \frac{3\pi}{2}$.

Example 4.2. *Consider the system*

$$\dot{x} = y + x^3 + xy^2, \qquad \dot{y} = -x + y^3 + x^2 y. \tag{15}$$

The ODE system (15) *is of the form* (1) *with polynomials that vanish at the origin of degree at least* 2 *given by* $A(x, y) = x^3 + xy^2$ *and* $B(x, y) = x^2y + y^3$. *The linearised system at the equilibrium point* $(0, 0)$ *has the eigenvalues* $\lambda = \pm i$. *We describe local stability using the two results mentioned in the section title of Section* 4. *The phase portrait for the system* (15) *is an unstable spiral as shown in Figure* 2.

(*i*) *Lyapunov's second method* (*Theorem* 2.1)*:*
Consider the Lyapunov function $V(x, y) = x^2 + y^2$ *where* $(x, y) \in \Re^2$.
Now $\frac{d}{dt}(V) = \partial_x(V(x, y)) \cdot \dot{x} + \partial_y(V(x, y)) \cdot \dot{y} = 2(x^4 + 2x^2y^2 + y^4) > 0$ *for all* $(x, y) \neq (0, 0)$. *From Theorem* 2.1 *and the derivative of* $V(x, y)$ *with respect to* t *being positive, it implies that the value of* V *increases along the trajectories near the origin thus the equilibrium point* $(0, 0)$ *is an unstable spiral.*

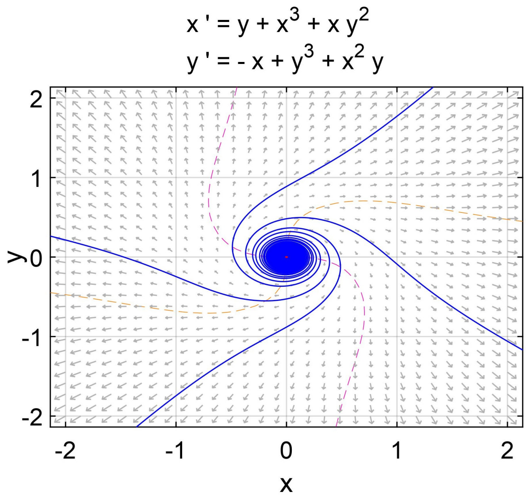

Figure 2: Phase portrait for system (15).

(*ii*) *Our method(Theorem 3.1):*
We let $x = r \cdot \cos\theta$ and $y = r \cdot \sin\theta$. In this case, the expansion of the ODE-system (15) yield $\alpha_1(\theta) = \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta$ and $\int_0^{2\pi} \alpha_1(\theta)d\theta = 2\pi > 0$. The equilibrium point $(0, 0)$ therefore is an unstable spiral since in the local stability condition of Theorem 3.1 the value of $a = -2\pi$ is less than zero.

Example 4.3. *Consider the system*

$$\dot{x} = y + x(x^2 + y^4), \qquad \dot{y} = -x + y(x^2 + y^4). \tag{16}$$

The ODE system (16) is of the form (1) with polynomials that vanish at the origin of degree at least 2 given by $A(x, y) = x^3 + xy^4$ and $B(x, y) = x^2y + y^5$. The linearised system at the equilibrium point $(0, 0)$ has the eigenvalues $\lambda = \pm i$. We describe local stability using the two results thus our result and Lyapunov stability result. The phase portrait for the system (16) is an unstable spiral as shown in Figure 3.

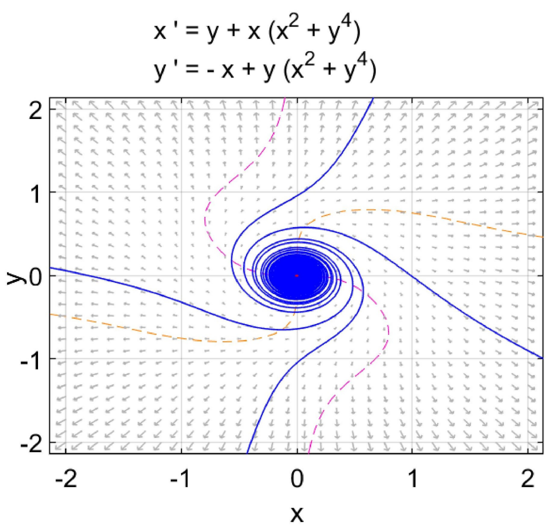

Figure 3: Phase portrait for system (16).

(i) *Lyapunov's second Method (Theorem 2.1):*
Consider the Lyapunov function $V(x,y) = \frac{1}{2}(x^2+y^2)$ *where* $x, y \in \Re^2$.
Now $\frac{d}{dt}(V) = \partial_x(V(x,y)) \cdot \dot{x} + \partial_y(V(x,y)) \cdot \dot{y} = (x^2+y^2)(x^2+y^4) > 0$ *for all* $(x,y) \neq (0,0)$.
From Theorem 2.1 and the fact that the derivative of $V(x,y)$ *with respect to* t *being positive, it implies that the value of* V *increases along the trajectories thus the equilibrium point* $(0,0)$ *is an unstable spiral.*

(ii) *Our method (Theorem 3.1):*
We let $x = r \cdot \cos\theta$ *and* $y = r \cdot \sin\theta$. *Here, the expansion of the system (16) gives* $\alpha_1(\theta) = \cos^4\theta + \cos^2\theta\sin^2\theta$ *and* $\int_0^{2\pi} \alpha_1(\theta)d\theta = \pi > 0$. *The equilibrium point* $(0,0)$ *therefore is an unstable spiral since in the local stability condition of Theorem 3.1 the value of* $a = -\pi$ *is less than zero.*

5 Boundedness of the Solutions to the System of Differential Equation (1)

In this section we give a definition of boundedness of a solution to a general system of differential equations and some result(s) on the general boundedness of solutions to such a general system of differential equations as one of the form (1). We use a result on boundedness of solutions re-stated herein (Theorem 5.1) to determine the boundedness of the solutions to the chosen examples in Section 4. We note that the technique used here yield the same boundedness results as those technique(s) in [17, 16] and [18] for this particular system considered. The following definition of boundedness of the solution and results on boundedness are adopted from [3, 11].

Definition 5.1. *We say that a given solution* $x(t, t_0, \phi)$ *to a general system of differential equations similar to (1) is bounded if for* $t \geq t_0$ *it satisfies*

$$\|x(t, t_0, \phi)\| \leq K(\|\phi\|, t_0),$$

where K *is a constant that depends on* t_0 *and* ϕ *is a given continuous and bounded initial function. We say that the solution(s) of the system is(are) uniformly bounded if* K *only depends on* ϕ.

Theorem 5.1. (Theorem 2.6 in [12]) *Suppose there exists a continuously differentiable Lyapunov functional* $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ *that satisfies*

$$\lambda_1\|x\|^p \leq V(t,x), \ v(t,x) \neq 0 \text{ if } x \neq 0,$$

and

$$V'(t,x) \leq -\lambda_2(t)V^q(t,x),$$

for some positive constants $\lambda_1, p, q > 1$ *where* $\lambda_2(t)$ *is a positive continuous function such that* $c_1 = \inf_{t \geq t_0 \geq 0} \lambda_2(t) > 0$. *Then all solutions* $x(t)$ *of the general system similar to the one in (1) are bounded as in Definition 5.1.*

Example 5.1. *Consider the system*

$$\dot{x} = y - x^3, \qquad \dot{y} = -x - y^3. \tag{17}$$

The ODE system (17) is in the form (1) with homogeneous polynomials that vanish at the origin of degree 3 given by $A(x,y) = -x^3$ *and* $B(x,y) = -y^3$. *We check the boundedness of the solutions to (17) using Theorem 5.1, consider the Lyapunov function* $V(x,y) = x^2 + y^2$ *where* $(x,y) \in \Re^2$.

Now $\frac{d}{dt}(V) = \partial_x(V(x,y)) \cdot \dot{x} + \partial_y(V(x,y)) \cdot \dot{y} = -2(x^4+y^4) \leq -2V^2(x,y)$ *for all* $(x,y) \neq (0,0)$ *and* $V(x,y) \neq 0$. *From Theorem 5.1 it implies the solutions are bounded as all the hypothesis in the theorem are satisfied.*

Example 5.2. *Consider the system*

$$\dot{x} = y + x^3 + xy^2, \qquad \dot{y} = -x + y^3 + x^2 y. \tag{18}$$

The ODE system (18) *is of the form* (1) *with polynomials that vanish at the origin of degree at least* 2 *given by* $A(x,y) = x^3 + xy^2$ *and* $B(x,y) = x^2y + y^3$. *Consider the Lyapunov function* $V(x,y) = x^2 + y^2 \neq 0$ *for* $(x,y) \neq (0,0)$ *where* $(x,y) \in \Re^2$.

Now $\frac{d}{dt}(V) = \partial_x(V(x,y)) \cdot \dot{x} + \partial_y(V(x,y)) \cdot \dot{y} = 2(x^4 + 2x^2y^2 + y^4) = 2(V^2(x,y)) > 0$ *for all* $(x,y) \neq (0,0)$. *This does not satisfy the hypotheses in Theorem* 5.1 *therefore the solution is not bounded.*

6 Discussions and Conclusions

Our main result shows that the equilibrium point $(0,0)$ is both a center and a focus for the system (1) if the local stability condition in the Theorem 3.1 is satisfied that is the value of the integral $\int_0^{2\pi} \alpha_1(\theta)d\theta$ being negative. This is because from (13) it means that there exists a positive constant C such that $r(2\pi) \leq r(0) - \epsilon_\star^3 \cdot a + C(\epsilon_\star^4)$ provided that $\epsilon_\star$ is sufficiently small. This therefore implies that the solution $r(\theta)$ tends to zero following a spiral as the angular variable θ continues to make turns around the trivial equilibrium point $(0,0)$. To illustrate further we have a strictly decreasing sequence $\{\epsilon_n\}_n^\infty$ for $r(t)$ as an integer multiple of π where $\epsilon_n \leq \epsilon_n(1 - a\epsilon_n^2) + C\epsilon_n^4$ for all natural numbers.

Our result compares well with the established Lyapunov's second method for stability Theorem 2.1 as demonstrated in Section 4. However, we do not know for now whether this result or a modification of such an integral can extend to other cases of two dimensional polynomial differential sytem where the eigenvalues of the linearised system is different from $\pm\omega i$ and the differential polynomial functions are of all degrees.

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Conflicts of Interest: The authors declare no conflict of interest.

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